

QUANTITATIVE BT-THEOREM AND AUTOMATIC CONTINUITY FOR STANDARD VON NEUMANN ALGEBRAS

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Dedicated to Professor G. A. Elliott on his 70th birthday

ABSTRACT. We prove a general criterion for a von Neumann algebra M in order to be in standard form. It is formulated in terms of an everywhere defined, invertible, antilinear, a priori not necessarily bounded operator, intertwining M with its commutant M' and acting as the $*$ -operation on the centre. We also prove a generalized version of the BT-Theorem which enables us to see that such an intertwiner must be necessarily bounded. It is shown that this extension of the BT-Theorem leads to the automatic boundedness of quite general operators which intertwine the identity map of a von Neumann algebra with a general bounded, real linear, operator valued map. We apply the last result to the automatic boundedness of linear operators implementing algebraic morphisms of a von Neumann algebra onto some Banach algebra, and to the structure of a W^* -algebra M endowed with a normal, semi-finite, faithful weight φ , whose left ideal \mathfrak{N}_φ admits an algebraic complement in the GNS representation space H_φ , invariant under the canonical action of M .

1. INTRODUCTION

A von Neumann algebra M on a Hilbert space H is usually called *standard* if there exists a bijective isometrical antilinear involution (called *conjugation*) $J : H \rightarrow H$ such that the mapping $x \mapsto Jx^*J$ is a $*$ -anti-isomorphism of M onto its commutant M' , acting as the $*$ -operation on the centre $Z(M)$ of M . In other words, J should satisfy the conditions

$$JMJ = M' \text{ and } JzJ = z^*, \quad z \in Z(M).$$

Any two $*$ -isomorphic standard von Neumann algebras are spatially isomorphic (see e.g. [7], Chapitre III, §1, Théorème 6 or [21], Corollary 10.15).

On the other hand, the Tomita-Takesaki Theory ([23]) enabled the construction of a standard representation of every von Neumann algebra. Indeed, each von Neumann algebra M has a normal semi-finite faithful (n.s.f. for short) weight φ , and then the associated GNS representation $\pi_\varphi : M \rightarrow B(H_\varphi)$ yields a $*$ -isomorphism of M onto the von Neumann algebra $\pi_\varphi(M)$, which is standard because the modular conjugation J_φ corresponding to φ satisfies

$$J_\varphi \pi_\varphi(M) J_\varphi = \pi_\varphi(M)' \text{ and } J_\varphi z J_\varphi = z^*, \quad z \in Z(\pi_\varphi(M))$$

(see e.g. [21], 10.14).

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We recall that a σ -finite von Neumann algebra is standard if and only if it has a cyclic and separating vector (see e.g. [21], 10.6, Corollary 1 in 10.13 and Theorem 10.25).

In conclusion, every von Neumann algebra has a standard representation which is unique up to spatial isomorphism. A more refined classification of the possible standard representations was elaborated by H. Araki ([1]) and A. Connes ([6]) for von Neumann algebras having a cyclic and separating vector, and by U. Haagerup ([11]) in the general case.

We shall prove a general criterion of standardness, namely that a von Neumann algebra M on a Hilbert space is standard whenever there exists a bijective antilinear operator $T : H \rightarrow H$ such that

$$TMT^{-1} = M' \text{ and } TzT^{-1} = z^*, z \in Z(M)$$

(Theorem 2.4). This theorem will be used in a forthcoming paper on tensor products of von Neumann algebras over von Neumann subalgebras.

Furthermore, we shall prove that the above antilinear operator T , not a priori assumed to be bounded, is necessarily bounded (Proposition 3.3). This boundedness result, which can be proved by using a result of E. L. Griffin (see [10], Theorems 1 and 2), arises also as particular case of a general automatic continuity theorem (Theorem 5.1), whose proof cannot be carried out by applying the results of Griffin. Its proof is based on a generalization of the classical BT-Theorem (Theorem 4.1), which could be of interest also elsewhere as it is shown in Section 6 containing some useful applications.

We shall use the terminology of [21]. In particular,

- $(\cdot | \cdot)$ will denote the inner product of a Hilbert space and it will be assumed linear in the first variable and antilinear in the second variable;
- $B(H)$ will denote the algebra of all bounded linear operators on the Hilbert space H , with the identity simply denoted by 1;
- $Z(M)$ will denote the centre of a von Neumann algebra $M \subset B(H)$;
- $l(x)$ and $r(x)$ will stay for the left and right support-projection of an operator x in some von Neumann algebra $M \subset B(H)$, where $l(x) = r(x) =: s(x)$ if x is normal, $z(x)$ for the central support projection of $x \in M$, and $s(\varphi)$ for the support projection of a normal positive linear functional φ on M .

2. A GENERAL CRITERION FOR STANDARDNESS OF VON NEUMANN ALGEBRAS

We say that a projection e in a von Neumann algebra M is *piecewise σ -finite* if there exists a family $(p_\iota)_{\iota \in I}$ of mutually orthogonal central projections of M such that $\sum_{\iota \in I} p_\iota = 1$ and all projections ep_ι are σ -finite. If the unit of M is piecewise σ -finite, then we call the von Neumann algebra M *piecewise σ -finite*.

Let us first consider the characterization of the standardness of piecewise σ -finite von Neumann algebras.

Proposition 2.1. *Let $M \subset B(H)$ be a piecewise σ -finite von Neumann algebra. If there is a bijective antilinear operator $T : H \rightarrow H$ such that*

$$TMT^{-1} = M' \text{ and } TzT^{-1} = z^*, \quad z \in Z(M),$$

then M is a standard von Neumann algebra.

Proof. Since T is commuting with the central projections of M and direct sums of standard von Neumann algebras are standard, we may assume without loss of generality that M is σ -finite.

According to Lemma 7.18 of [21], there exists a projection $p \in Z(M)$ such that the reduced/induced algebra $M_p = \{x \mid pH : pH \longrightarrow pH; x \in M\} \subset B(pH)$ has a cyclic vector $\xi \in pH$, while $M_{1-p} \subset B((1-p)H)$ has a separating vector $\eta \in (1-p)H$.

Then $T^{-1}\xi \in pH$ is separating for M_p . Namely, suppose that $xT^{-1}\xi = 0$ for some $x \in pM$. Then

$$yTxT^{-1}\xi = 0, \quad y \in pM.$$

This means

$$0 = yTxT^{-1}\xi = TxT^{-1}y\xi, \quad y \in pM,$$

and by the cyclicity of ξ for M_p we deduce that

$$TxT^{-1} = TxpT^{-1} = TxT^{-1}p = 0.$$

Thus, $x = 0$.

Using now the Dixmier–Maréchal Theorem (see [8], Corollaire 1), we infer that pH contains a vector which is cyclic and separating for M_p .

Similarly, the vector $T\eta \in (1-p)H$ is cyclic for M_{1-p} . Indeed, the orthogonal projection e' onto the closure of $(1-p)MT\eta = MT(1-p)\eta = MT\eta$ belongs to M' and $e' \leq 1-p$. Furthermore, $e'T\eta = T\eta$ and so $T^{-1}e'T\eta = \eta$, where $T^{-1}e'T \in (1-p)M$. Since η is separating for M_{1-p} , we get that $T^{-1}e'T = 1-p$, hence $e' = 1-p$. In other words $(1-p)MT\eta$ is dense in $(1-p)H$.

Applying again the Dixmier–Maréchal Theorem, we infer that $(1-p)H$ contains a vector which is cyclic and separating for M_{1-p} .

We conclude that M has a cyclic and separating vector and therefore it is a standard von Neumann algebra. □

Since finite von Neumann algebras are piecewise σ -finite (see e.g. [21], Lemma 7.2), Proposition 2.1 implies immediately :

Corollary 2.2. *Let $M \subset B(H)$ be a finite von Neumann algebra such that for an appropriate bijective antilinear operator $T : H \longrightarrow H$ we have*

$$TMT^{-1} = M' \text{ and } TzT^{-1} = z^*, \quad z \in Z(M).$$

Then M is a standard von Neumann algebra. □

Now we are going to characterize the standard properly infinite von Neumann algebras.

We recall that, for a given infinite cardinal γ , a properly infinite von Neumann algebra M is called *uniform of type γ* if there exists a family $(e_\iota)_{\iota \in I}$ of equivalent, mutually orthogonal, piecewise σ -finite projections in M such that $\sum_{\iota \in I} e_\iota = 1$ and

the cardinality of I is γ . Every properly infinite von Neumann algebra M has a unique decomposition in uniform components: there exists a set Γ of distinct cardinals and a family $(p_\gamma)_{\gamma \in \Gamma}$ of non-zero central projections, uniquely determined by the conditions

$$\sum_{\gamma \in \Gamma} p_\gamma = 1, \quad M_{p_\gamma} \text{ is uniform of type } \gamma \text{ for every } \gamma \in \Gamma$$

(see [21], proposition 8.5).

Proposition 2.3. *Let $M \subset B(H)$ be a properly infinite von Neumann algebra. If there exists a multiplicative antilinear isomorphism $\theta : M \longrightarrow M'$, which acts on the centre of M as the $*$ -operation, then M is a standard von Neumann algebra.*

Proof. First we reduce the proof to the case when θ is additionally a $*$ -map.

Let φ be some n.s.f. weight on M , $\pi_\varphi : M \longrightarrow B(H_\varphi)$ the associated GNS representation, and J_φ the corresponding modular conjugation. Then $\theta_\varphi : \pi_\varphi(M) \ni \pi_\varphi(x) \longmapsto J_\varphi \pi_\varphi(x) J_\varphi \in \pi_\varphi(M)'$ is a multiplicative antilinear isomorphism commuting with the $*$ -operation.

Since $\theta_\varphi \circ \pi_\varphi \circ \theta^{-1} : M' \longrightarrow \pi_\varphi(M)'$ is an algebra isomorphism, by Theorem I in [15] there exists an invertible $0 \leq a' \in M'$ such that

$$M' \ni x' \longmapsto (\theta_\varphi \circ \pi_\varphi \circ \theta^{-1})(a' x' a'^{-1}) \in \pi_\varphi(M)'$$

is a $*$ -isomorphism. Consequently its composition with $(\theta_\varphi \circ \pi_\varphi)^{-1}$, that is

$$M' \ni x' \longmapsto \theta^{-1}(a' x' a'^{-1}) \in M,$$

as well as the inverse map

$$\theta_o : M \ni x \longmapsto a'^{-1} \theta(x) a' \in M'$$

of this, are multiplicative antilinear isomorphisms commuting with the $*$ -operation.

Thus θ_o is completely additive, positive, and preserves Murray-von Neumann equivalence of projections. Moreover, θ_o acts on the centre of M as the $*$ -operation.

Taking now in account the decomposability of M in uniform components, as well as the fact that direct sums of standard von Neumann algebras are standard, we can assume in the sequel without loss of generality that M is uniform of type γ for some infinite cardinal γ .

Since $M' = \theta_o(M)$, the commutant M' is uniform of type γ . On the other hand, taking into account that

$$M \ni x \longmapsto J_\varphi \pi_\varphi(x) J_\varphi \in \pi_\varphi(M)'$$

is a multiplicative antilinear isomorphism commuting with the $*$ -operation, also the commutant $\pi_\varphi(M)'$ is uniform of type γ .

Using now a classical implementation theorem (essentially Theorem 2 of [9], cf. [21], Theorem 8.6), we conclude that the $*$ -isomorphism $\pi_\varphi : M \longrightarrow \pi_\varphi(M)$ of the von Neumann algebras M and $\pi_\varphi(M)$, whose commutants are uniform of the same type γ , is spatial. Since any von Neumann algebra which is spatially isomorphic to a standard von Neumann algebra is still standard, we infer that M is a standard von Neumann algebra. □

A direct consequence of Corollary 2.2 and Proposition 2.3 is the following general standardness criterion :

Theorem 2.4. (General criterion for standardness of von Neumann algebras) *Let M be a von Neumann algebra on a Hilbert space H . M is acting in standard form if and only if there is a bijective antilinear operator $T : H \longrightarrow H$ such that*

$$TMT^{-1} = M' \text{ and } TzT^{-1} = z^*, \quad z \in Z(M).$$

□

3. LINEAR OPERATORS COMMUTING WITH A VON NEUMANN ALGEBRA

As it is shown in the Appendix, an everywhere defined, bijective, antilinear operator on a Hilbert space, even an involutive one, might be unbounded. Nevertheless, as we shall show in this section, this is not the case for an operator T satisfying the assumptions in Theorem 2.4.

A first proof will be based on the result of E. L. Griffin (see [10], Theorems 1 and 2) reported below:

Theorem 3.1. *Let M be a von Neumann algebra on a Hilbert space H . In order that every linear operator $T : H \rightarrow H$ satisfying*

$$Tx = xT, \quad x \in M$$

be bounded, it is necessary and sufficient that no minimal projection p of $Z(M)$ exists with pM finite-dimensional and pM' infinite-dimensional.

□

From Theorem 3.1 it follows immediately the next automatic boundedness result:

Corollary 3.2. *If $M \subset B(H)$ is a standard von Neumann algebra, then every linear operator $T : H \rightarrow H$ satisfying*

$$Tx = xT, \quad x \in M$$

is bounded.

Proof. By Theorem 3.1 it is enough to verify that if $p \in Z(M)$ is a projection such that the reduced/induced von Neumann algebra M_p is finite-dimensional, then also its commutant $(M')_p$ is finite-dimensional.

But since M is standard and the projection p is central, also M_p is a standard von Neumann algebra. In particular, $(M')_p$ is $*$ -anti-isomorphic to M_p and therefore it is finite-dimensional.

□

An alternative proof, based on an extended version of the BT-Theorem, will be presented in Corollary 5.3.

Now we are ready to prove

Proposition 3.3. *If $M \subset B(H)$ is a von Neumann algebra and $T : H \rightarrow H$ is a bijective antilinear operator such that*

$$TMT^{-1} = M' \text{ and } TzT^{-1} = z^*, \quad z \in Z(M),$$

then T must be bounded.

Proof. First at all, by Theorem 2.4 the von Neumann algebra M is standard. Let $J : H \rightarrow H$ be a conjugation satisfying the conditions

$$JMJ = M' \text{ and } JzJ = z^*, \quad z \in Z(M).$$

Then $JT : H \rightarrow H$ is a bijective linear operator such that the mapping

$$M \ni x \mapsto JT x (JT)^{-1} = JT x T^{-1} J \in M$$

is an algebra automorphism.

Next, by Theorem I in [15], there exists an invertible $0 \leq a \in M$ such that

$$M \ni x \mapsto JTa x (JT a)^{-1} = JTa x a^{-1} T^{-1} J \in M$$

is a $*$ -isomorphism.

Finally, since every $*$ -isomorphism between standard von Neumann algebras is spatial (see e.g. [7], Chapitre III, §1, Théorème 6 or [21], Corollary 10.15), there exists a unitary $U \in B(H)$ for which

$$JTax(JTa)^{-1} = U^{-1}xU, \quad x \in M,$$

that is

$$(UJTa)x = x(UJTa), \quad x \in M.$$

Now Corollary 3.2 yields the boundedness of $UJTa$, hence also the boundedness of $T = JU^{-1}(UJTa)a^{-1}$. \square

4. A QUANTITATIVE BT-THEOREM

The classical "BT-Theorem" of Murray and von Neumann (see e.g. [19], Theorem 2.7.14 or [21], C.6.1) states that if M is a von Neumann algebra on the Hilbert space H , $\xi_o \in H$ and ξ belongs to the closure of $M\xi_o$, then $\xi = bT\xi_o$ where $b \in M$ and T is a densely defined, closed linear operator, affiliated to M . Thus, roughly speaking, we can "lift" any vector in $\overline{M\xi_o}$ to an operator bT "related" to M . The more recent proof, due essentially to R. V. Kadison and presented by C. F. Skau in [20], Lemma 3.4, can be extended to obtain the following "quantitative" version of the BT-Theorem, which will allow to "lift" vector sequences in $\overline{M\xi_o}$ converging sufficiently fast to zero in operator sequences which converge to zero in operator norm.

Theorem 4.1. (Quantitative BT-Theorem) *Let M be a von Neumann algebra on a Hilbert space H , $\xi_o \in H$, $(\xi_k)_{k \geq 1}$ a sequence in $\overline{M\xi_o}$, and $(\gamma_k)_{k \geq 1}$ a sequence in $(0, +\infty)$ such that*

$$\sum_{k=1}^{\infty} \frac{1}{\gamma_k} \|\xi_k\|^2 < +\infty.$$

Then there exist $a \in M$ with $0 \leq a \leq 1$, $\eta_o \in (\overline{aH}) \cap (\overline{M\xi_o}) = \overline{aM\xi_o}$, as well as a sequence $(b_k)_{k \geq 1}$ in the operator norm closure of Ma , such that

$$\begin{aligned} a\eta_o &= \xi_o, \\ b_k\eta_o &= \xi_k \text{ and } \|b_k\| \leq \sqrt{\gamma_k}, \quad k \geq 1. \end{aligned}$$

Proof. Let $k \geq 1$ be arbitrary. Since $\xi_k \in \overline{M\xi_o}$, we can find by induction a sequence $(x_{k,j})_{j \geq 0}$ in M such that

$$\left\| \xi_k - \sum_{j=0}^n x_{k,j} \xi_o \right\| \leq \frac{1}{4^{n+2}} \|\xi_k\|, \quad n \geq 0.$$

Then

$$\begin{aligned} \|x_{k,o} \xi_o\| &= \|\xi_k - (\xi_k - x_{k,o} \xi_o)\| \leq \frac{17}{16} \|\xi_k\|, \\ \|x_{k,n} \xi_o\| &= \left\| \left(\xi_k - \sum_{j=0}^{n-1} x_{k,j} \xi_o \right) - \left(\xi_k - \sum_{j=0}^n x_{k,j} \xi_o \right) \right\| \leq \frac{5}{4^{n+2}} \|\xi_k\|, \quad n \geq 1. \end{aligned}$$

In particular, we can write

$$\xi_k = \sum_{j=0}^{\infty} x_{k,j} \xi_o \quad (4.1)$$

where the series converges in norm.

Now let us define

$$y_p = \left(1 + \sum_{k=1}^p \sum_{j=0}^p \frac{4^{j+1}}{\gamma_k} x_{k,j}^* x_{k,j} \right)^{1/2} \in M, \quad p \geq 1.$$

Since the square root function \sqrt{t} is operator increasing on $[0, +\infty)$ (see e.g. [16], Proposition 1.3.8 or [22], Proposition 2.7), we have

$$1 \leq y_1 \leq y_2 \leq \dots,$$

and since $1/t$ is operator decreasing on $(0, +\infty)$ (see e.g. [16], Proposition 1.3.6 or [22], 2.6 (7)), we have

$$1 \geq y_1^{-1} \geq y_2^{-1} \geq \dots \geq 0.$$

Therefore, the sequence $(y_p^{-1})_{p \geq 1}$ is convergent in the strong operator topology to some $a \in M, 0 \leq a \leq 1$.

On the other hand, we have for every $p \geq 1$

$$\begin{aligned} \|y_p \xi_o\|^2 &= (y_p^2 \xi_o | \xi_o) = \left(\xi_o + \sum_{k=1}^p \sum_{j=0}^p \frac{4^{j+1}}{\gamma_k} x_{k,j}^* x_{k,j} \xi_o \middle| \xi_o \right) \\ &= \|\xi_o\|^2 + \sum_{k=0}^p \sum_{j=0}^p \frac{4^{j+1}}{\gamma_k} \|x_{k,j} \xi_o\|^2 \\ &= \|\xi_o\|^2 + \sum_{k=0}^p \frac{4}{\gamma_k} \|x_{k,o} \xi_o\|^2 + \sum_{k=1}^p \sum_{j=1}^p \frac{4^{j+1}}{\gamma_k} \|x_{k,j} \xi_o\|^2 \\ &\leq \|\xi_o\|^2 + \sum_{k=0}^p \frac{4}{\gamma_k} \cdot \frac{17^2}{16^2} \|\xi_k\|^2 + \sum_{k=1}^p \sum_{j=1}^p \frac{4^{j+1}}{\gamma_k} \cdot \frac{25}{4^{2j+4}} \|\xi_k\|^2 \\ &= \|\xi_o\|^2 + \sum_{k=0}^p \frac{1}{\gamma_k} \|\xi_k\|^2 \left(\frac{289}{64} + \sum_{j=1}^p \frac{25}{4^{j+3}} \right) \\ &\leq \|\xi_o\|^2 + \sum_{k=0}^p \frac{1}{\gamma_k} \|\xi_k\|^2 \left(\frac{289}{64} + \sum_{j=1}^{\infty} \frac{25}{4^{j+3}} \right) \\ &= \|\xi_o\|^2 + \frac{223}{48} \sum_{k=0}^p \frac{1}{\gamma_k} \|\xi_k\|^2 \leq \|\xi_o\|^2 + 5 \sum_{k=0}^{\infty} \frac{1}{\gamma_k} \|\xi_k\|^2, \end{aligned}$$

that is

$$\|y_p \xi_o\| \leq c, \quad p \geq 1$$

where

$$c = \left(\|\xi_o\|^2 + 5 \sum_{k=0}^{\infty} \frac{1}{\gamma_k} \|\xi_k\|^2 \right)^{1/2} < +\infty.$$

Since closed balls in H are weakly compact, there exists a weakly convergent subnet $(y_{p_i} \xi_o)_{i \in \mathbb{N}}$ of the bounded sequence $(y_p \xi_o)_{p \geq 1}$ (actually there exists a weakly convergent subsequence because the closed balls in the closed linear span of the sequence

$(y_p \xi_o)_{p \geq 1}$ are compact and metrizable). Let η denote the weak limit of $(y_{p_i} \xi_o)_i$. Clearly, $\eta \in \overline{M \xi_o}$.

We claim that $a\eta = \xi_o$. Indeed, for every $\xi \in H$ we have

$$\begin{aligned} |(y_{p_i} \xi_o | a\xi) - (\xi_o | \xi)| &= |(y_{p_i} \xi_o | a\xi) - (y_{p_i} \xi_o | y_{p_i}^{-1} \xi)| = |(y_{p_i} \xi_o | a\xi - y_{p_i}^{-1} \xi)| \\ &\leq \|y_{p_i} \xi_o\| \cdot \|a\xi - y_{p_i}^{-1} \xi\| \leq c \|a\xi - y_{p_i}^{-1} \xi\| \end{aligned}$$

and $y_p^{-1} \xrightarrow{so} a$ yields $\lim_i (y_{p_i} \xi_o | a\xi) = (\xi_o | \xi)$. Taking now into account that the weak limit of $(y_{p_i} \xi_o)_i$ is η , we conclude that $(\eta | a\xi) = (\xi_o | \xi)$, hence

$$(a\eta | \xi) = (\eta | a\xi) = (\xi_o | \xi).$$

Let η_o denote the orthogonal projection of η onto \overline{aH} , that is $\eta_o = s(a)\eta$ where $s(a) \in M$ is the support projection of a . Then $a\eta_o = as(a)\eta = a\eta = \xi_o$.

Denoting now by p'_{ξ_o} the cyclic projection in M' associated to ξ_o , that is the orthogonal projection onto $\overline{M \xi_o}$, we have $\eta = p'_{\xi_o} \eta$ and consequently

$$\eta_o = s(a)\eta = s(a)p'_{\xi_o} \eta = p'_{\xi_o} s(a)\eta \in \overline{M \xi_o}.$$

Therefore $\eta_o \in (\overline{aH}) \cap (\overline{M \xi_o})$.

The inclusion $(\overline{aH}) \cap (\overline{M \xi_o}) \supset \overline{aM \xi_o}$ is obvious. For the proof of the converse inclusion let $\zeta \in (\overline{aH}) \cap (\overline{M \xi_o})$ be arbitrary. Choosing a sequence $(\eta_k)_{k \geq 1}$ in H such that $\zeta = \lim_k a\eta_k$, we have

$$\zeta = p'_{\xi_o} \zeta = \lim_k p'_{\xi_o} a\eta_k = \lim_k a p'_{\xi_o} \eta_k.$$

Since $p'_{\xi_o} \eta_k \in \overline{M \xi_o}$, $k \geq 1$, it follows that ζ belongs to the closure of $\overline{aM \xi_o}$, that is to $\overline{aM \xi_o}$.

Summing up the above, we have $a \in M$, $0 \leq a \leq 1$, and $\eta_o \in (\overline{aH}) \cap (\overline{M \xi_o}) = \overline{aM \xi_o}$ such that

$$a\eta_o = \xi_o. \quad (4.2)$$

Let now $k \geq 1$ and $j \geq 0$ be arbitrary. For every $p \geq \max(k, j)$ we have

$$\begin{aligned} y_p^{-1} \left(\frac{4^{j+1}}{\gamma_k} x_{k,j}^* x_{k,j} \right) y_p^{-1} &\leq y_p^{-1} \left(\sum_{k'=1}^p \sum_{j'=0}^p \frac{4^{j+1}}{\gamma_{k'}} x_{k',j'}^* x_{k',j'} \right)^{1/2} y_p^{-1} \\ &= y_p^{-1} (y_p^2 - 1) y_p^{-1} = 1 - y_p^{-2} \\ &\leq 1. \end{aligned}$$

Taking into account that $y_p^{-1} \xrightarrow{so} a$, we obtain

$$a \left(\frac{4^{j+1}}{\gamma_k} x_{k,j}^* x_{k,j} \right) a \leq 1 \iff a x_{k,j}^* x_{k,j} a \leq \frac{\gamma_k}{4^{j+1}} \iff \|x_{k,j} a\| \leq \frac{\sqrt{\gamma_k}}{2^{j+1}}.$$

By the above estimation, we can define

$$b_k = \sum_{j=0}^{\infty} x_{k,j} a, \quad k \geq 1$$

where the series converges in the norm and thus b_k belongs to the operator norm closure of Ma . Moreover,

$$\|b_k\| \leq \sum_{j=0}^{\infty} \|x_{k,j} a\| \leq \sum_{j=0}^{\infty} \frac{\sqrt{\gamma_k}}{2^{j+1}} \leq \sqrt{\gamma_k}, \quad k \geq 1.$$

By (4.2) and (4.1) holds also

$$b_k \eta_o = \sum_{j=0}^{\infty} x_{k,j} a \eta_o = \sum_{j=0}^{\infty} x_{k,j} \xi_o = \xi_k, \quad k \geq 1.$$

□

Now we show how the above theorem can be used to "lift" vector sequences which converge sufficiently fastly to zero in operator sequences which converge to zero in operator norm.

Corollary 4.2. (BT-Theorem for convergence) *Let M be a von Neumann algebra on a Hilbert space H , $\xi_o \in H$, and $(\xi_k)_{k \geq 1}$ a sequence in $\overline{M\xi_o}$ such that*

$$\sum_{k=1}^{\infty} \|\xi_k\|^2 < +\infty.$$

Then there exist $a \in M$ with $0 \leq a \leq 1$, $\eta_o \in (\overline{aH}) \cap (\overline{M\xi_o}) = \overline{aM\xi_o}$, as well as a sequence $(b_k)_{k \geq 1}$ in the operator norm closure of Ma satisfying

$$\begin{aligned} a \eta_o &= \xi_o, \\ b_k \eta_o &= \xi_k \text{ for } k \geq 1, \\ \lim_{k \rightarrow \infty} \|b_k\| &= 0. \end{aligned}$$

Proof. It is well known that for any convergent series $\sum_{k=0}^{\infty} \alpha_k$ of positive numbers there is a sequence $0 < \gamma_k \rightarrow 0$ such that the series $\sum_{k=0}^{\infty} \frac{1}{\gamma_k} \alpha_k$ is still convergent (see e.g. [14], §39, 175.4 or [5], Lemma 1.5): we can take, for example,

$$\gamma_k = \begin{cases} \left(\sum_{j=k}^{\infty} \alpha_j \right)^{1/2} + \left(\sum_{j=k+1}^{\infty} \alpha_j \right)^{1/2} & \text{if } \alpha_k > 0, \\ \frac{1}{2^k} & \text{if } \alpha_k = 0. \end{cases}$$

Applying the above remark to the series $\sum_{k=0}^{\infty} \|\xi_k\|^2$ we get a sequence $0 < \gamma_k \rightarrow 0$ such that

$$\sum_{k=1}^{\infty} \frac{1}{\gamma_k} \|\xi_k\|^2 < +\infty.$$

Now we can apply Theorem 4.1 obtaining a, η_o and the sequence $(b_k)_{k \geq 1}$ having the desired properties.

□

We note that the classical BT-Theorem follows with $\xi_k = 0$, identically for $k \geq 2$.

5. AUTOMATIC CONTINUITY PROPERTIES OF STANDARD VON NEUMANN ALGEBRAS

Let M be a standard von Neumann algebra on a Hilbert space H . Corollary 3.2 claims the boundedness of every linear operator on H which commutes with all operators belonging to M , that is which intertwines the identity map on M with itself. Using Corollary 4.2 we shall next prove a general continuity theorem, which

implies the continuity of additive maps intertwining the identity map on M with an arbitrary bounded real linear map from M into the bounded linear operators on some Banach space X .

We recall (see e.g. [24], Problem 5-3-103) : if X, Y are topological vector spaces and $T : X \rightarrow Y$ is a \mathbb{Q} -homogeneous mapping which has closed graph, then T is \mathbb{R} -homogeneous. Indeed, for any $x \in X$ and real λ , choosing a sequence $(\lambda_k)_{k \geq 1}$ of rational numbers converging to λ , we have

$$(\lambda_k x, T(\lambda_k x)) = (\lambda_k x, \lambda_k T(x)) \rightarrow (\lambda x, \lambda T(x)),$$

so $(\lambda x, \lambda T(x))$ belongs to the graph of T .

Consequently, if X, Y are Banach spaces and $T : X \rightarrow Y$ is an additive, hence \mathbb{Q} -linear map, having closed graph, then T is real linear and by the closed graph theorem it follows also its boundedness.

Theorem 5.1. (Boundedness of intertwining operators) *Let M be a von Neumann algebra on a Hilbert space H . Then (i) \implies (ii) \implies (iii) where :*

- (i) *The weak $*$ topology on M' coincides with the weak operator topology, that is every normal positive linear functional on M' is a finite sum of functionals of the form $\omega'_\zeta : M' \ni x' \mapsto (x' \zeta | \zeta)$, $\zeta \in H$.*
- (ii) *For every sequence $(\xi_k)_{k \geq 1}$ in H , there exist $n \geq 1$ and $\zeta_1, \dots, \zeta_n \in H$ such that ξ_k belongs to the closure of $\sum_{j=1}^n M \zeta_j$ for each $k \geq 1$.*
- (iii) *For any Banach space X , bounded real linear map Φ of M into the Banach space $B(X)$ of all bounded linear operators on X , and additive operators $T_1, T_2 : H \rightarrow X$ satisfying the intertwining condition*

$$T_1 x = \Phi(x) T_2, \quad x \in M, \tag{5.1}$$

the operator T_1 and the composition of T_2 with the canonical map of X onto the quotient Banach space $X / \bigcap_{x \in M} \text{Ker } \Phi(x)$ (identifiable with $\bigvee_{x \in M} r(\Phi(x)) T_2$ if X is a Hilbert space) are both necessarily real linear and bounded.

Proof. To show (i) \implies (ii) let us assume that (i) holds and let $(\xi_k)_{k \geq 1}$ be a sequence in H .

The support projection $s(\omega_{\xi'_k})$ of $\omega_{\xi'_k} : M' \ni x' \mapsto (x' \xi_k | \xi_k)$ is the orthogonal projection onto $\overline{M \xi_k}$, so the range of the support projection $s(\varphi')$ of the normal positive linear functional

$$\varphi' = \sum_{k=1}^{\infty} \frac{1}{2^k(1 + \|\xi_k\|^2)} \omega_{\xi'_k}$$

on M' , which is $\bigvee_{k=1}^{\infty} s(\omega_{\xi'_k})$, contains the sequence $(\xi_k)_{k \geq 1}$. But by (i) there exist

finitely many $\zeta_1, \dots, \zeta_n \in H$ such that $\varphi' = \sum_{j=1}^n \omega_{\zeta'_j}$ and so $s(\varphi') = \bigvee_{j=1}^n s(\omega_{\zeta'_j})$ is the

orthogonal projection onto the closure of $\sum_{j=1}^n M \zeta_j$.

For (ii) \implies (iii) let us assume that (ii) holds and let X be a Banach space, $\Phi : M \longrightarrow B(X)$ a bounded real linear map, and $T_1, T_2 : H \longrightarrow X$ additive maps satisfying the intertwining condition (5.1).

To prove the continuity of T_1 it is enough to verify its continuity in 0, which follows once we prove that for every sequence $(\xi_k)_{k \geq 1}$ in H with

$$\|\xi_k\| \leq \frac{1}{2^k}, \quad k \geq 1$$

the convergence $T_1 \xi_k \longrightarrow 0$ holds true.

By (ii) there exist finitely many $\zeta_1, \dots, \zeta_n \in H$ such that each ξ_k belongs to the closure of $\sum_{j=1}^n M \zeta_j$.

Let $M_n(M)$ denote the von Neumann algebra of all $n \times n$ matrices with entries in M , acting on the Hilbert space $H_n = \bigoplus_{j=1}^n H$. With

$$\tilde{\zeta} = \bigoplus_{j=1}^n \zeta_j \in H_n, \quad \tilde{\xi}_k = \xi_k \oplus \underbrace{0 \oplus \dots \oplus 0}_{n-1 \text{ times}} \in H_n, \quad k \geq 1$$

we have $\sum_{k=1}^{\infty} \|\tilde{\xi}_k\|^2 = \sum_{k=1}^{\infty} \|\xi_k\|^2 < +\infty$ and $\tilde{\xi}_k \in \overline{M_n(M) \tilde{\zeta}}, k \geq 1$, so we can apply Corollary 4.2 obtaining (among other things) a vector $\tilde{\eta} \in H_n$ and a sequence $(\tilde{b}_k)_{k \geq 1}$ in $M_n(M)$ with

$$\begin{aligned} \tilde{b}_k \tilde{\eta} &= \tilde{\xi}_k, \quad k \geq 1, \\ \lim_{k \rightarrow \infty} \|\tilde{b}_k\| &= 0. \end{aligned}$$

If b_{k1}, \dots, b_{kn} is the first row of the matrix \tilde{b}_k and η_1, \dots, η_n are the components of $\tilde{\eta}$ then

$$\begin{aligned} \xi_k &= \sum_{j=1}^n b_{kj} \eta_j, \quad k \geq 1, \\ \|b_{kj}\| &\leq \|\tilde{b}_k\|, \quad k \geq 1 \text{ and } 1 \leq j \leq n. \end{aligned}$$

Consequently

$$\begin{aligned} \|T_1 \xi_k\| &= \left\| \sum_{j=1}^n T_1 b_{kj} \eta_j \right\| = \left\| \sum_{j=1}^n \Phi(b_{kj}) T_2 \eta_j \right\| \leq \|\Phi\| \sum_{j=1}^n \|b_{kj}\| \cdot \|T_2 \eta_j\| \\ &\leq \|\Phi\| \|\tilde{b}_k\| \sum_{j=1}^n \|T_2 \eta_j\| \longrightarrow 0 \end{aligned}$$

and so $T_1 \xi_k \longrightarrow 0$.

Having verified the continuity of the additive, hence \mathbb{Q} -linear map T_1 , its real linearity follows immediately.

We go next to prove that the graph of the composition $\widehat{T_2}$ of T_2 with the canonical map of X onto the quotient Banach space $X / \bigcap_{x \in M} \text{Ker} \Phi(x)$ is closed.

We shall denote the canonical image of $\zeta \in X$ by $\widehat{\zeta}$, so that $\widehat{T_2}(\xi) = \widehat{T_2(\xi)}, \xi \in H$. Taking into account the additivity of T_2 , it is enough to prove that if $(\xi_k)_{k \geq 1} \subset H$

is a sequence such that $\xi_k \rightarrow 0$ and $\widehat{T_2\xi_k} \rightarrow \widehat{\zeta}$, then $\widehat{\zeta} = \widehat{0}$, that is

$$\Phi(x)\zeta = 0, \quad x \in M. \quad (5.2)$$

For let $x \in M$ be arbitrary. If φ is any bounded linear functional on X , then the composition $\varphi \circ \Phi(x)$ is a bounded linear functional on X which vanishes on $\bigcap_{y \in M} \text{Ker } \Phi(y)$, defining thus the bounded linear functional

$$\widehat{\varphi \circ \Phi(x)} : X / \bigcap_{x \in M} \text{Ker } \Phi(x) \ni \widehat{\eta} \mapsto \varphi(\Phi(x)\eta).$$

Since by (5.1) and by the boundedness of T_1 , we have

$$\begin{aligned} \varphi(\Phi(x)\zeta) &= (\widehat{\varphi \circ \Phi(x)})(\widehat{\zeta}) = \lim_{k \rightarrow \infty} (\widehat{\varphi \circ \Phi(x)})(\widehat{T_2\xi_k}) \\ &= \lim_{k \rightarrow \infty} \varphi(\Phi(x)T_2\xi_k) = \lim_{k \rightarrow \infty} \varphi(T_1x\xi_k) \\ &= 0 \end{aligned}$$

for any φ , the Hahn-Banach theorem yields (5.2).

We conclude that the additive, hence \mathbb{Q} -linear map $\widehat{T_2}$ has closed graph and, by the remarks before the statement of the theorem, it follows its real-linearity and continuity. □

In particular :

Corollary 5.2. *Let M be a von Neumann algebra on a Hilbert space H such that every normal positive linear functional on M' is a vector functional, that is of the form $\omega'_\xi : M' \ni x' \mapsto (x'\xi|\xi)$ for an appropriate $\xi \in H$. Then statement (iii) in Theorem 5.1 holds true.* □

Corollary 5.3. (Cases of automatic boundedness of intertwining operators) *Let M be a von Neumann algebra on a Hilbert space H . Then statement (iii) in Theorem 5.1 holds true in each one of the following situations :*

- (1) M has a cyclic vector;
- (2) M is properly infinite;
- (3) M is standard.

Proof. (1) is an immediate consequence of Theorem 5.1.

(2) and (3) follow from Corollary 5.2 because every normal positive linear form on the commutant of a properly infinite von Neumann algebra is of the form ω'_ξ for some $\xi \in H$ (see e.g. [21], Theorem 8.16) and standard von Neumann algebras have the same property (see [1], Theorem 6, [6], Théorème 2.7, [11], Lemma 2.10). □

Remark 5.1. *With $X = H$, $\Phi = \text{id}$ and $T_1 = T_2$, statement (iii) in Theorem 5.1 reduces to the automatic continuity of any additive map $T : H \rightarrow H$ satisfying*

$$Tx = xT, \quad x \in M.$$

Therefore Corollary 5.3 can be used to obtain an alternative proof of Corollary 3.2. We point out the fact that the more general situation in Theorem 5.1 may not be treated by the method of [10].

6. SOME APPLICATIONS

The present section is devoted to provide some applications of the previous result (cf. Theorem 5.1) on the boundedness of intertwining operators.

We first start with the algebra homomorphism $x \mapsto TxT^{-1}$ implemented in a canonical way by an invertible, a priori non necessarily bounded, linear operator.

Proposition 6.1. *Let M be a von Neumann algebra on a Hilbert space H , and T a bijective linear map from H onto a Banach space X such that $\{TxT^{-1}; x \in M\}$ is a closed subalgebra of the Banach algebra $B(X)$ of all bounded linear operators on X . Assuming that M satisfies one of the conditions of Corollary 5.3, the operator T must be necessarily bounded.*

Proof. Let us denote by Φ the injective algebra homomorphism

$$M \ni x \longmapsto TxT^{-1} \in B(X).$$

By our assumption $\Phi(M)$ is a closed, hence complete subalgebra of $B(X)$, and Φ is an algebra isomorphism of M onto it. Since the von Neumann algebra M is semi-simple and semi-simplicity is an algebraic invariant (see e.g. [4], Chapter III, § 24, Definition 13), $\Phi(M)$ is a semi-simple algebra endowed with the two complete norms:

$$\Phi(x) \longmapsto \|\Phi(x)\| \text{ and } \Phi(x) \longmapsto \|x\|.$$

By a classical theorem of B. E. Johnson ([12], see also [4], Chapter III, § 25, Theorem 9, a short proof was done in [3]), these norms should be equivalent and therefore the map $\Phi : M \rightarrow B(X)$ is bounded.

Applying now Corollary 5.3, we end the proof. \square

Now let φ be a n.s.f. weight on a W^* -algebra M . We consider the left ideal

$$\mathfrak{N}_\varphi = \{x \in M; \varphi(x^*x) < +\infty\}$$

and denote by $\pi_\varphi : M \rightarrow B(H_\varphi)$ the associated GNS representation. x_φ will stay for $x \in \mathfrak{N}_\varphi$ considered an element of the Hilbert space H_φ and we shall discuss the existence of an invariant algebraic complement of \mathfrak{N}_φ in H_φ .

For τ a n.s.f. trace and A a (possibly unbounded) positive, self-adjoint linear operator in H_τ , affiliated with $\pi_\tau(M)$, we shall use the notation of Pedersen-Takesaki $\tau(A \cdot)$ for the normal, semi-finite weight defined by

$$\begin{aligned} \tau(Ab) &:= \lim_{k \rightarrow \infty} \tau\left(\pi_\tau^{-1}(A^{1/2}\chi_{[0,k]}(A)) \cdot b \cdot \pi_\tau^{-1}(A^{1/2}\chi_{[0,k]}(A))\right) \\ &= \lim_{k \rightarrow \infty} \tau\left(b^{1/2}\pi_\tau^{-1}(A\chi_{[0,k]}(A))b^{1/2}\right), \quad 0 \leq b \in M, \end{aligned}$$

where $\chi_{[0,k]}$ stands for the characteristic function of $[0, k]$ (see [17], Paragraph 4). The weight $\tau(A \cdot)$ is faithful if and only if A is injective and $\tau(A \cdot) \geq \lambda \tau$ for some scalar $\lambda > 0$ if and only if $A \geq \lambda$.

Proposition 6.2. *Let $M \neq \{0\}$ be a W^* -algebra equipped with a n.s.f. weight φ , and \mathfrak{X} a left ideal of M , contained in \mathfrak{N}_φ and such that $X = \{x_\varphi; x \in \mathfrak{X}\}$ is dense in H_φ . Then the following statements are equivalent :*

- (i) *X admits an algebraic complement in H_φ which is invariant under the action of $\pi_\varphi(M)$.*
- (ii) *$\mathfrak{X} = \mathfrak{N}_\varphi$ and $\{x_\varphi; x \in \mathfrak{N}_\varphi\} = H_\varphi$.*

- (iii) $\mathfrak{X} = \mathfrak{N}_\varphi$ and $\sup_{\substack{0 \leq b \in M \\ b \neq 0}} \frac{\|b\|}{\varphi(b)} < +\infty$.
- (iv) $\mathfrak{X} = \mathfrak{N}_\varphi$ and the reduced W^* -algebra eMe is finite-dimensional for every projection $e \in M$ with $\varphi(e) < +\infty$.
- (v) $\mathfrak{X} = \mathfrak{N}_\varphi$, M is the direct product of a family $(M_\iota)_{\iota \in I}$ of type I factors and, denoting by τ_ι the canonical trace on M_ι (i.e. the n.s.f. trace which is equal to 1 in every minimal projection), for some scalar $\lambda > 0$ we have

$$\varphi(b) \geq \lambda \left(\bigoplus_{\iota \in I} \tau_\iota \right)(b), \quad 0 \leq b \in M.$$

Proof. The implication (ii) \implies (i) is trivial.

For (i) \implies (ii) let us assume that there exists a $\pi_\varphi(M)$ -invariant linear subspace Y of H_φ such that $H_\varphi = X \oplus Y$, where " \oplus " stands for inner algebraic direct sum. Let P denote the corresponding projection operator: $P : H_\varphi \longrightarrow H_\varphi$ is the linear operator defined by $P(\xi + \eta) = \xi$ where $\xi \in X, \eta \in Y$. It is easily seen that P satisfies the commutation condition

$$P\pi_\varphi(x) = \pi_\varphi(x)P, \quad x \in M$$

and therefore, by Corollary 3.2 (or by Corollary 5.3), we infer that P is a bounded operator. Consequently, the dense linear subspace $X = PH_\varphi$ of H_φ is also closed, hence $X = H_\varphi$. Now

$$H_\varphi = X \subset \{x_\varphi; x \in \mathfrak{N}_\varphi\} \subset H_\varphi$$

means $X = \{x_\varphi; x \in \mathfrak{N}_\varphi\} \iff \mathfrak{X} = \mathfrak{N}_\varphi$ and $\{x_\varphi; x \in \mathfrak{N}_\varphi\} = H_\varphi$, that is (ii).

For the proof of equivalence (ii) \iff (iii) we shall use the fact that

$$\begin{aligned} \{(\xi, x); x \in \mathfrak{N}_\varphi, \xi = x_\varphi\} &\text{ is a closed subset of } H_\varphi \times M \\ &\text{ with respect to the product of the norm-topologies} \end{aligned} \quad (6.1)$$

(in other words the linear operator $\mathfrak{N}_\varphi \ni x \longmapsto x_\varphi \in H_\varphi$ is closed with respect to the norm-topologies; cf. with the Lebesgue continuity property considered in [25], Section 2 and [2], Section 2.4).

To verify (6.1) let us assume that $\xi \in H_\varphi, x \in M$ are such that, for some sequence $(x_k)_{k \geq 1}$ in \mathfrak{N}_φ , we have $\|\xi - (x_k)_\varphi\| \longrightarrow 0$ and $\|x - x_k\| \longrightarrow 0$. Then by the weak* lower semicontinuity of the normal weight φ , we have

$$\varphi(x^*x) \leq \varliminf_{k \rightarrow \infty} \varphi(x_k^*x_k) = \varliminf_{k \rightarrow \infty} \|(x_k)_\varphi\|^2 = \|\xi\|^2 < +\infty.$$

In particular $x \in \mathfrak{N}_\varphi$, and for every $y \in \mathfrak{N}_\varphi$ we obtain :

$$\begin{aligned} J_\varphi \pi_\varphi(y) J_\varphi x_\varphi &= \pi_\varphi(x) J_\varphi y_\varphi = \lim_{k \rightarrow \infty} \pi_\varphi(x_k) J_\varphi y_\varphi \\ &= \lim_{k \rightarrow \infty} J_\varphi \pi_\varphi(y) J_\varphi (x_k)_\varphi = J_\varphi \pi_\varphi(y) J_\varphi \xi. \end{aligned}$$

Taking into account that the identity operator on H_φ belongs to the weak operator closure of $J_\varphi \pi_\varphi(\mathfrak{N}_\varphi) J_\varphi$, we conclude that $x_\varphi = \xi$.

Now (ii) \implies (iii) follows by a simple application of the closed graph theorem. Indeed, if $\{x_\varphi; x \in \mathfrak{N}_\varphi\} = H_\varphi$ then we can consider the everywhere defined linear operator

$$H_\varphi = \{x_\varphi; x \in \mathfrak{N}_\varphi\} \ni x_\varphi \longmapsto x \in M$$

whose graph is closed by (6.1). According to the closed graph theorem, there exists a constant $c \geq 0$ such that $\|x\| \leq c \varphi(x^*x)^{1/2}$ holds true for all $x \in \mathfrak{N}_\varphi$. Consequently,

$$\frac{\|b\|}{\varphi(b)} = \left(\frac{\|b^{1/2}\|}{\varphi((b^{1/2})^*b^{1/2})^{1/2}} \right)^2 \leq c^2, \quad 0 \leq b \in M, 0 < \varphi(b) < +\infty.$$

Actually, the inequality $\frac{\|b\|}{\varphi(b)} \leq c^2$ holds for any non-zero $0 \leq b \in M$ because for $\varphi(b) = +\infty$ we have $\frac{\|b\|}{\varphi(b)} = \frac{\|b\|}{+\infty} = 0 \leq c^2$.

For the converse implication let us assume that (iii) holds and put

$$c := \left(\sup_{\substack{0 \leq b \in M \\ b \neq 0}} \frac{\|b\|}{\varphi(b)} \right)^{1/2} < +\infty.$$

Then

$$\|x\| = \|x^*x\|^{1/2} \leq \left(c^2 \varphi(x^*x) \right)^{1/2} = c \|x_\varphi\|, \quad x \in \mathfrak{N}_\varphi.$$

Let now $\xi \in H_\varphi$ be arbitrary and $(x_k)_{k \geq 1}$ a sequence in \mathfrak{N}_φ with $\|\xi - (x_k)_\varphi\| \rightarrow 0$. By the above inequality $(x_k)_{k \geq 1}$ is a Cauchy sequence with respect to the norm of M , so it is norm-convergent to some $x \in M$. According to (6.1), we conclude that $x \in \mathfrak{N}_\varphi$ and $\xi = x_\varphi$. This shows that (ii) holds.

Now we have the equivalences $(i) \iff (ii) \iff (iii)$. To complete the proof we shall show that $(ii) \implies (iv) \implies (v) \implies (iii)$.

For $(ii) \implies (iv)$ let us assume (ii) and let $e \in M$ be any projection satisfying $\varphi(e) < +\infty$. Since reflexive C^* -algebras are finite-dimensional (see the proof of Proposition 2 in [18] or [13], Chapter 10, Exercise 10.5.17 (iii)), it is enough to prove that every state ψ on eMe is normal.

Let $\tilde{\psi}$ be the extension of ψ to a state on M defined by $\tilde{\psi}(x) := \psi(exe)$. Let also denote by $\pi_{\tilde{\psi}} : M \rightarrow B(H_{\tilde{\psi}})$ the associated GNS representation and by $\xi_{\tilde{\psi}}$ its canonical cyclic vector, so that $\tilde{\psi}(x) = (\pi_{\tilde{\psi}}(x)\xi_{\tilde{\psi}} | \xi_{\tilde{\psi}})$.

The everywhere defined linear operator

$$T : H_\varphi = \{y_\varphi; y \in \mathfrak{N}_\varphi\} \ni y_\varphi \mapsto \pi_{\tilde{\psi}}(y)\xi_{\tilde{\psi}} \in H_{\tilde{\psi}}$$

satisfies the intertwining condition

$$T\pi_\varphi(x) = \pi_{\tilde{\psi}}(x)T, \quad x \in M.$$

Indeed, we have for every $x \in M$ and $y \in \mathfrak{N}_\varphi$,

$$T\pi_\varphi(x)y_\varphi = T(xy)_\varphi = \pi_{\tilde{\psi}}(xy)\xi_{\tilde{\psi}} = \pi_{\tilde{\psi}}(x)(\pi_{\tilde{\psi}}(y)\xi_{\tilde{\psi}}) = \pi_{\tilde{\psi}}(x)T y_\varphi.$$

Applying Corollary 5.3 we infer that T is bounded.

Now we are ready to prove the normalness of ψ . For let $(b_\kappa)_{\kappa \in K}$ be a bounded, increasing net of positive elements of eMe and $b := \sup_{\kappa \in K} b_\kappa$. By the normality of π_φ we have

$$(b_\kappa)_\varphi = (b_\kappa e)_\varphi = \pi_\varphi(b_\kappa)e_\varphi \rightarrow \pi_\varphi(b)e_\varphi = (be)_\varphi = b_\varphi$$

and the boundedness of T yields

$$\pi_{\tilde{\psi}}(b_\kappa)\xi_{\tilde{\psi}} = T(b_\kappa)_\varphi \rightarrow T b_\varphi = \pi_{\tilde{\psi}}(b)\xi_{\tilde{\psi}}.$$

Consequently,

$$\psi(b_\kappa) = \tilde{\psi}(b_\kappa) = (\pi_{\tilde{\psi}}(b_\kappa)\xi_{\tilde{\psi}}|\xi_{\tilde{\psi}}) \longrightarrow (\pi_{\tilde{\psi}}(b)\xi_{\tilde{\psi}}|\xi_{\tilde{\psi}}) = \tilde{\psi}(b) = \psi(b)$$

and we are done.

For $(iv) \implies (v)$ let us assume that statement (iv) holds.

First we show that then M is the direct product of a family $(M_\iota)_{\iota \in I}$ of type I factors. The proof consists in a straightforward application of the Zorn Lemma once we show that any non-zero central projection $q \in Z(M)$ majorizes some non-zero central projection p such that Mp is a type I factor.

For we notice that the restriction of φ to Mq is semi-finite, hence there exists a non-zero projection $e_o \in M, e_o \leq q$, having finite weight $\varphi(e_o) < +\infty$. According to (iv) the reduced W^* -algebra $e_o M e_o$ is finite-dimensional, so it contains a minimal projection $0 \neq e \leq e_o \leq q$.

According to the Zorn Lemma, there exists a maximal set $\mathcal{F} \supset \{e\}$ of mutually orthogonal projections in M , all equivalent to e . Since all projections in \mathcal{F} have the same central support $z(e) \leq q$, $\sum_{f \in \mathcal{F}} f$ is less or equal than $z(e)$. We claim that

actually $\sum_{f \in \mathcal{F}} f = z(e)$. Indeed, by the comparison theorem there exists a central projection $q_o \leq z(e)$ such that

$$\begin{aligned} eq_o &\prec \left(z(e) - \sum_{f \in \mathcal{F}} f \right) q_o, \\ e(z(e) - q_o) &\succ \left(z(e) - \sum_{f \in \mathcal{F}} f \right) (z(e) - q_o). \end{aligned} \tag{6.2}$$

If q_o were non-zero, by the minimality of e the projection $0 \neq eq_o \leq e$ would be equal to e and the first above relation in (6.2) would imply

$$e \prec \left(z(e) - \sum_{f \in \mathcal{F}} f \right) q_o \leq z(e) - \sum_{f \in \mathcal{F}} f,$$

contradicting the maximality of \mathcal{F} . Thus $q_o = 0$ and therefore the second relation in (6.2) yields

$$e \succ z(e) - \sum_{f \in \mathcal{F}} f.$$

If $z(e) - \sum_{f \in \mathcal{F}} f$ were $\neq 0$, then it would be equivalent to a non-zero subprojection of e , which by the minimality of e should be equal to e . But this would contradict the maximality of \mathcal{F} . Consequently $z(e) - \sum_{f \in \mathcal{F}} f = 0$.

By the above, $p := z(e)$ is a non-zero central subprojection of q , equal to the sum of the mutually orthogonal, equivalent, minimal projections belonging to \mathcal{F} . Then Mp is $*$ -isomorphic to the von Neumann algebra of all bounded linear operators on a Hilbert space of dimension $\text{card}(\mathcal{F})$ (see e.g. [21], Theorem 4.22), and thus is a type I factor.

Now, knowing that M is the direct product of a family $(M_\iota)_{\iota \in I}$ of type I factors, we can consider on each M_ι the canonical trace τ_ι , and then on M the n.s.f. trace $\tau := \bigoplus_{\iota \in I} \tau_\iota$. By the Radon-Nikodym type theorem of Pedersen and Takesaki ([17],

Theorem 5.12) there exists a positive, self-adjoint linear operator in H_τ , affiliated with $\pi_\tau(M)$, such that $\varphi = \tau(A \cdot)$. Using again (iv), we shall verify that 0 does not belong to the spectrum of A , that is $A \geq \lambda$ for some scalar $\lambda > 0$. Then it will follow $\varphi = \tau(A \cdot) \geq \lambda \tau$ and we can conclude that (v) holds.

To this end let us assume the contrary, that is that 0 belongs to the spectrum of A . Since $\varphi = \tau(A \cdot)$ is faithful, A must be injective, so 0 cannot be an isolated point of the spectrum of A . Therefore we can find real numbers

$$\alpha_1 > \beta_1 > \alpha_2 > \beta_2 > \dots > 0, \quad \alpha_k < 2^{-k} \text{ for all } k \geq 1,$$

such that every interval (α_k, β_k) intersects the spectrum of A . Thus the mutually orthogonal spectral projections $\chi_{(\alpha_k, \beta_k)}(A) \in \pi_\tau(M)$ are all non-zero. Choose for every $k \geq 1$ some minimal projection $f_k \in M$ less or equal than $\pi_\tau^{-1}(\chi_{(\alpha_k, \beta_k)}(A))$ and put $f_o := \sum_{k \geq 1} f_k$.

Then $\varphi(f_o) < +\infty$. Indeed, we have for every $k \geq 1$

$$\begin{aligned} \varphi(f_k) &= \varphi\left(\pi_\tau^{-1}(\chi_{(\alpha_k, \beta_k)}(A))f_k\right) = \tau\left(f_k \pi_\tau^{-1}(A \chi_{(\alpha_k, \beta_k)}(A))f_k\right) \\ &\leq \alpha_k \tau\left(f_k \pi_\tau^{-1}(\chi_{(\alpha_k, \beta_k)}(A))f_k\right) = \alpha_k \tau(f_k) = \alpha_k \\ &< 2^{-k}, \end{aligned}$$

hence, by the normalness of φ ,

$$\varphi(f_o) = \sum_{k \geq 1} \varphi(f_k) \leq \sum_{k \geq 1} 2^{-k} = 1 < +\infty.$$

By (iv) it should follow that the reduced algebra $f_o M f_o$ is finite-dimensional, But $f_o M f_o$ contains the infinitely many mutually orthogonal non-zero projections f_k and this contradiction shows that 0 cannot belong to the spectrum of A .

Finally, for (v) \implies (iii) let us assume that (v) holds. Since the Hilbert-Schmidt norm majorizes the operator norm, we have

$$\tau_\iota(x_\iota^* x_\iota)^{1/2} \geq \|x_\iota\|, \quad x_\iota \in M_\iota, \iota \in I.$$

Denoting $\tau := \bigoplus_{\iota \in I} \tau_\iota$, it follows for every $x = \prod_{\iota \in I} x_\iota \in M$

$$\tau(x^* x) = \sum_{\iota \in I} \tau_\iota(x_\iota^* x_\iota) \geq \sum_{\iota \in I} \|x_\iota\|^2 \geq \sup_{\iota \in I} \|x_\iota\|^2 = \|x\|^2 = \|x^* x\|.$$

Thus we obtain for every non-zero $0 \leq b \in M$ with $\varphi(b) < +\infty$:

$$\varphi(b) \geq \lambda \tau(b) \geq \lambda \|b\| \iff \frac{\|b\|}{\varphi(b)} \leq \lambda^{-1}.$$

Consequently, (iii) holds true. □

For bounded functionals, Proposition 6.2 entails :

Corollary 6.3. *Let $M \neq \{0\}$ be a W^* -algebra equipped with a faithful, normal state φ , and \mathfrak{X} a left ideal of M such that $X = \{x_\varphi; x \in \mathfrak{X}\}$ is dense in H_φ . Then the following statements are equivalent :*

- (i) *X admits an algebraic complement in H_φ which is invariant under the action of $\pi_\varphi(M)$.*

- (ii) $\mathfrak{X} = \mathfrak{N}_\varphi$ and $\{x_\varphi; x \in \mathfrak{N}_\varphi\} = H_\varphi$.
- (iii) $\mathfrak{X} = \mathfrak{N}_\varphi$ and $\sup_{\substack{0 \leq b \in M \\ b \neq 0}} \frac{\|b\|}{\varphi(b)} < +\infty$.
- (iv) $\mathfrak{X} = \mathfrak{N}_\varphi$ and M is finite-dimensional.

□

7. APPENDIX: SOME EXAMPLES

In the present appendix we show that there exist linear and antilinear involutions on a infinite dimensional Hilbert space, defined everywhere and unbounded.

Let H be an infinite dimensional Hilbert space equipped with a Hamel basis $(\xi_\iota)_{\iota \in I}$. Then each $\xi \in H$ can be uniquely written as

$$\xi = \sum_{\iota \in I} \lambda_\iota(\xi) \xi_\iota,$$

where all but finitely many coefficients $\lambda_\iota(\xi)$ vanish. In other words, the set F_ξ of all indices $\iota \in I$ with $\lambda_\iota(\xi) \neq 0$ is finite.

The linear functionals $(\lambda_\iota)_{\iota \in I}$ are called the *coordinate functionals* of the Hamel basis $(\xi_\iota)_{\iota \in I}$.

Proposition 7.1. *For a fixed Hamel basis $(\lambda_\iota)_{\iota \in I}$, $|\{\iota \in I \mid \lambda_\iota \text{ continuous}\}| < +\infty$.*

Proof. Suppose that $|\{\iota \in I \mid \lambda_\iota \text{ continuous}\}| = +\infty$. Without loss of generality, we can suppose that the Hamel basis under consideration is made of unit vectors. Choose a sequence $(\xi_{\iota_n})_{n \geq 1} \subset (\xi_\iota)_{\iota \in I}$ such that all the corresponding coordinate functions $(\lambda_{\iota_n})_{n \geq 1}$ are continuous. The sum

$$\xi := \sum_{n=1}^{+\infty} \frac{1}{2^n} \xi_{\iota_n}$$

is a well defined element of H and

$$\sum_{n=1}^{+\infty} \frac{1}{2^n} \xi_{\iota_n} = \xi = \sum_{\iota \in F_\xi} \lambda_\iota(\xi) \xi_\iota.$$

By the finiteness of F_ξ there exists $n_o \geq 1$ such that

$$\{\iota_{n_o}, \iota_{n_o+1}, \dots\} \cap F_\xi = \emptyset.$$

Now the sequence

$$\eta_k := \sum_{n=1}^k \frac{1}{2^n} \xi_{\iota_n} - \xi = \sum_{n=1}^k \frac{1}{2^n} \xi_{\iota_n} - \sum_{n=1}^{+\infty} \frac{1}{2^n} \xi_{\iota_n} = - \sum_{n=k+1}^{+\infty} \frac{1}{2^n} \xi_{\iota_n}$$

is clearly convergent to 0 and by the continuity of $\lambda_{\iota_{n_o}}$ we obtain

$$\lim_{k \rightarrow \infty} \lambda_{\iota_{n_o}}(\eta_k) = \lambda_{\iota_{n_o}}\left(\lim_{k \rightarrow \infty} \eta_k\right) = \lambda_{\iota_{n_o}}(0) = 0. \quad (7.1)$$

But for $k > n_o$ we have

$$\lambda_{\iota_{n_o}}(\eta_k) = \sum_{n=1}^k \frac{1}{2^n} \lambda_{\iota_{n_o}}(\xi_{\iota_n}) - \lambda_{\iota_{n_o}}(\xi) = \frac{1}{2^{n_o}}$$

because $\iota_{n_o} \notin F_\xi$. Consequently

$$\lim_{k \rightarrow \infty} \lambda_{\iota_{n_o}}(\eta_k) = \frac{1}{2^{n_o}},$$

in contradiction to (7.1). \square

It is immediate to argue that, for each $n \geq 1$, any infinite-dimensional Hilbert space H admits a Hamel basis with at least n continuous coordinate functions and infinitely many non continuous coordinate functions. It is indeed enough to split H in the orthogonal sum of an n -dimensional linear subspace and its orthogonal complement, which is infinite dimensional, and apply Proposition 7.1 to the latter.

Proposition 7.2. *Let H be an infinite dimensional Hilbert space. Then there exists an unbounded linear or antilinear involution (i.e. with square equal to the identity operator on H) $C : H \rightarrow H$.*

Proof. As explained below, there exists a Hamel basis $(\xi_\iota)_{\iota \in I}$ of H admitting a continuous coordinate function λ_{ι_1} , and a non continuous one λ_{ι_2} , for some ι_1, ι_2 in I .

Define the linear operator $C_1 : H \rightarrow H$ by

$$C_1 \xi := \sum_{\{\iota \in I \mid \iota \neq \iota_1, \iota_2\}} \lambda_\iota(\xi) \xi_\iota + \lambda_{\iota_2}(\xi) \xi_{\iota_1} + \lambda_{\iota_1}(\xi) \xi_{\iota_2}. \quad (7.2)$$

If C_1 were continuous, then the linear functional defined by

$$f(\xi) := \frac{(\xi - C_1 \xi \mid \xi_{\iota_1} - \xi_{\iota_2})}{\|\xi_{\iota_1} - \xi_{\iota_2}\|^2} = \lambda_{\iota_1}(x) - \lambda_{\iota_2}(x)$$

would also be continuous, but it is not. Thus C_1 is the linear involution we were searching for.

Concerning the antilinear case, either the antilinear operator $C_2 : H \rightarrow H$ defined by

$$C_2 \xi := \sum_{\iota \in I} \overline{\lambda_\iota(\xi)} \xi_\iota \quad (7.3)$$

is not continuous and we are done, or it is continuous. In the latter, the operator

$$C := C_1 C_2,$$

where C_1 is the non continuous linear involution given by (7.2), while C_2 is the continuous antilinear involution given by (7.3), is antilinear and non continuous. It is also an involution as $C_1 C_2 = C_2 C_1$. \square

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